Formulae for Seasonal Effects and some Asymptotic Values

Maury Goodman and Phil Schreiner - NuMI note 3844 November 27, 2007

1 definitions

1. $\eta = 0.054$, K factor. This is not the π/K ratio, but it has the π/K ratio in it.
2. $\epsilon_\pi = 115$ GeV. Critical energy in the atmosphere for which π interaction and π decay are equal
3. $\epsilon_K = 850$ GeV. Critical energy in the atmosphere for which K interaction and K decay are equal.
4. f_π. Fraction of pions which are positive.
5. f_K. Fraction of Kaons which are positive.
6. $\gamma = 1.7$. Spectral index (not 2.7!)

2 Charge Ratio asymptotia

For practice, the Schreiner equation is:

$$r = \frac{\left\{ \frac{f_\pi}{1+\frac{1.1E_\mu \cos\theta}{115 \text{ GeV}}} + \frac{\eta f_K}{1+\frac{1.1E_\mu \cos\theta}{850 \text{ GeV}}} \right\}}{\left\{ \frac{1-f_\pi}{1+\frac{1.1E_\mu \cos\theta}{115 \text{ GeV}}} + \frac{\eta(1-f_K)K}{1+\frac{1.1E_\mu \cos\theta}{850 \text{ GeV}}} \right\}} \tag{1}$$

This has asymptotes at low and high $E_\mu^{surface} \cos\theta_z$.

$$R_{low} = \frac{f_\pi + \eta f_k}{1 - f_\pi + \eta(1 - f_K)} \tag{2}$$

and

$$R_{high} = \frac{\epsilon_\pi f_\pi + \eta\epsilon_K f_K}{\epsilon_\pi(1 - f_\pi) + \eta\epsilon_K(1 - f_K)} \tag{3}$$

3 Seasonal coefficient asymptotia

Eric's Equation 34 for the α coefficient which includes effect from both π and K is

$$\alpha = < 1/[\frac{\gamma}{\gamma + 1} \, \frac{\epsilon_\pi + \eta\epsilon_K}{1.1E_\mu^{surface} \cos\theta_z} + 1 + \eta] > \tag{4}$$

I now compare the effect of including Kaons. I call the equation with Kaons $\alpha_{K\pi}$ and the equation without Kaons α_π. Low slant depth corresponds to a low $E_\mu^{surface}\cos\theta_z$ threshold, while high slant depth corresponds to a high $E_\mu^{surface}\cos\theta_z$ threshold. At low $E_\mu^{surface}\cos\theta_z$:

$$\frac{\alpha_{K\pi}}{\alpha_\pi} = \frac{1}{1+\eta\frac{\epsilon_K}{\epsilon_\pi}} \tag{5}$$

At high $E_\mu^{surface}\cos\theta_z$:

$$\frac{\alpha_{K\pi}}{\alpha_\pi} = \frac{1}{1+\eta} \tag{6}$$

4 discussion

Eric's formula for the temperature effective increases with $E_\mu^{surface}\cos\theta_z$ (and hence with slant depth) and Kaons reduces the effect. But the change due to Kaons seems to increase at low $E_\mu^{surface}\cos\theta_z$. This is counter intuitive to many of us, because we think of effects from Kaons as contributing more strongly at high $E_\mu^{surface}\cos\theta_z$. Equation 6 shows a 5% effective which gets amplified at low energy by 850/115. This is plotted in Figure 1.

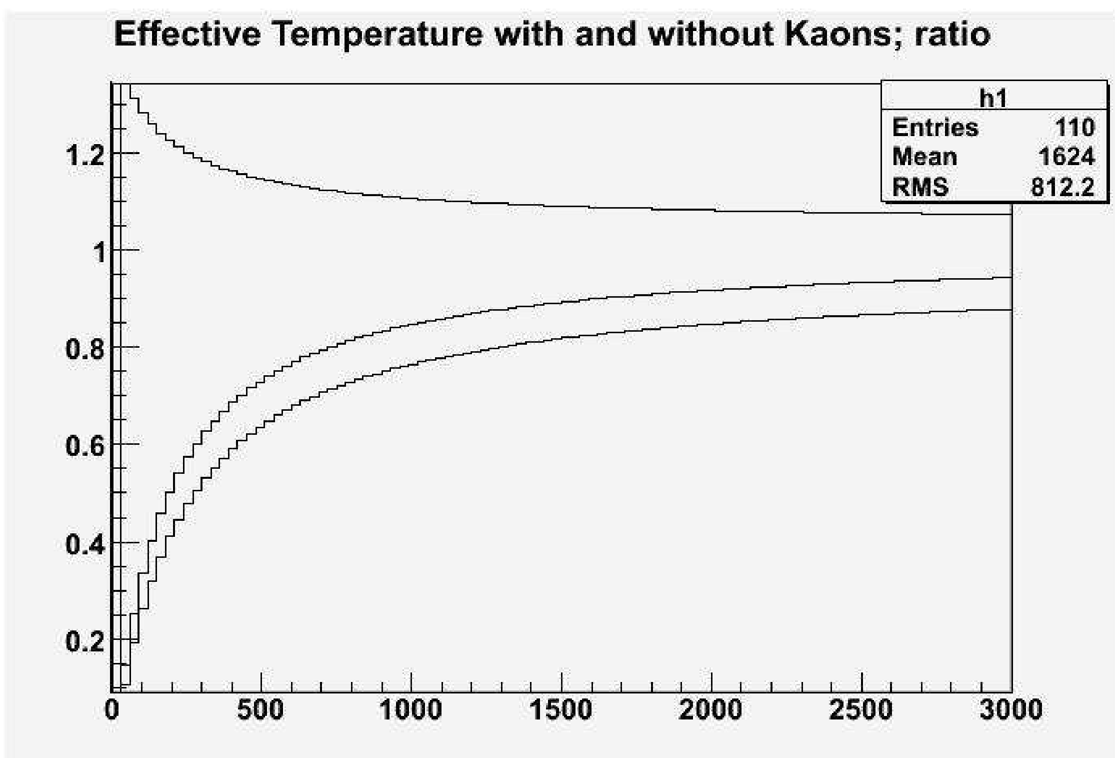

Figure 1: The middle curve is $\alpha(X)$ for pions only. The lower curve includes Kaons. The upper curve is the ratio.

5 Derivation of Formula for alpha-T with Kaons

I will keep our term for eta and leave the 1.1 as a number. We start with Eric's formulae in A11 and A25:

$$I_\mu \propto E^{-\gamma}\left(\frac{1}{\gamma+(\gamma+1)1.1E\cos\theta/\varepsilon_\pi}+\frac{\eta}{\gamma+(\gamma+1)1.1E\cos\theta/\varepsilon_K}\right) \tag{7}$$

and

$$\alpha_T = -\frac{E}{I_\mu}\frac{\partial I_\mu}{\partial E} - \gamma \tag{8}$$

Let's define the two terms in the Denominator of Equation 7 as D_π and D_K. Taking the derivative of Equation 7 gets us four terms. Since we divide it out, I'll ignore the constant from the start

$$\frac{\gamma \times E^{-(\gamma+1)}}{D_\pi}+\frac{E^{-\gamma}(\gamma+1)1.1\cos\theta/\varepsilon_\pi}{D_\pi^2}+\frac{\gamma\times\eta\times E^{-(\gamma+1)}}{D_K}+\frac{\eta\times E^{-\gamma}(\gamma+1)1.1\cos\theta/\varepsilon_K}{D_K^2} \tag{9}$$

We then multiply the 2nd and 4th terms by E/E so we bring out a single power term, and we multiply the 1st term by D_π/D_π and the 3rd term by D_K/D_K:

$$\begin{aligned}\frac{\partial I_\mu}{\partial E} = E^{-(\gamma+1)}\times \quad & ([\gamma(\gamma+(\gamma+1)1.1E\cos\theta/\varepsilon_\pi)+(\gamma+1)1.1E\cos\theta/\varepsilon_\pi]/D_\pi^2 \\ & +\eta[\gamma(\gamma+(\gamma+1)1.1E\cos\theta/\varepsilon_K)+(\gamma+1)1.1E\cos\theta/\varepsilon_K]/D_K^2)\end{aligned} \tag{10}$$

Plugging into equation 8, we get

$$\alpha = \frac{[\gamma^2+(1+\gamma)^2 1.1E\cos\theta/\varepsilon_\pi]*D_K/D_\pi+\eta[\gamma^2+(1+\gamma)^2 1.1E\cos\theta/\varepsilon_K]*D_\pi/D_K}{D_K+\eta D_\pi}-\gamma \tag{11}$$

Which leads to our not-too-simple expression for the temperature coefficient:

$$\alpha = \frac{\frac{\gamma^2+(\gamma+1)^2 1.1E\cos\theta/\varepsilon_\pi}{(\gamma+(\gamma+1)1.1E\cos\theta/\varepsilon_\pi)^2}+\eta\frac{\gamma^2+(\gamma+1)^2 1.1E\cos\theta/\varepsilon_K}{(\gamma+(\gamma+1)1.1E\cos\theta/\varepsilon_K)^2}}{\frac{1}{\gamma+(\gamma+1)1.1E\cos\theta/\varepsilon_\pi}+\frac{\eta}{\gamma+(\gamma+1)1.1E\cos\theta/\varepsilon_K}}-\gamma \tag{12}$$

This is not the same as equation 4. Both formulae are plotted in Figure 2.

6 Seasonal coefficient asymptotia with new formula

With Equation 12 we can repeat the exercise of looking at asymptotic values of α at low and high energy. A plot is made in blue triangles in Figure 3 of $\alpha_{K\pi}/\alpha_\pi$. It is everywhere below unity. Our faith in our intuition is restored!

Looking at the asymptotes; at low $E_\mu^{surface}\cos\theta_z$: $\alpha_{K\pi}(0)=0$, $\alpha_\pi(0)=0$ and $\alpha_{K\pi}/\alpha_\pi = 1$. At high $E_\mu^{surface}\cos\theta_z$: $\alpha_{K\pi}(0)=1$, $\alpha_\pi(0)=1$ and $\alpha_{K\pi}/\alpha_\pi=1$.

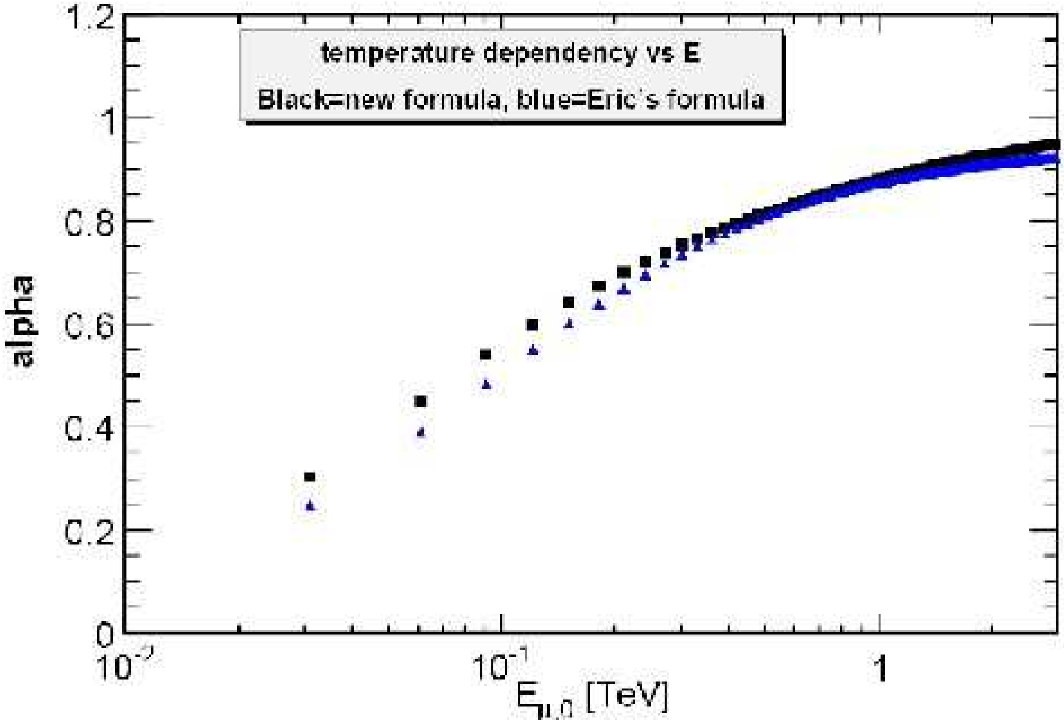

Figure 2: Difference between two calculations of alpha with Kaons.

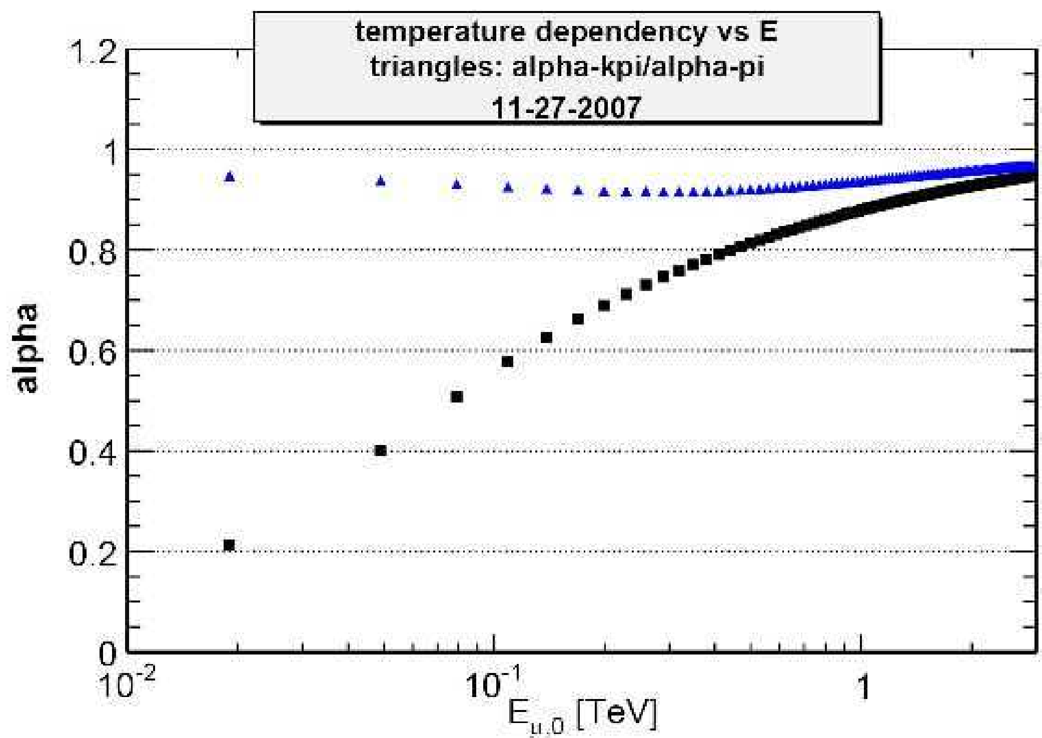

Figure 3: Temperature dependency versus E (Black Squares) and the ratio $\alpha_{K\pi}/\alpha_{\pi}$ versus E (Blue Triangles).